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Generalized Faddeev Equations for the Many-Cluster Problem

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Abstract

In the framework of the Faddeev formalism, the cluster description of nuclei is developed. A set of generalized Faddeev equations are thus obtained to apply to many-cluster problem. The connectedness and the solvability of the obtained integral equations are considered. The usefulness of introducing some approximations is discussed. Numerical calculations of our model for practical nuclei are in good agreement with experimental measurement.

1. *Introduction*

Faddeev (1961a, 1961b, 1963) introduce an exact and correct theory for the scattering amplitude in treating the three-body problem. The obtained Faddeev integral equations have no δ functions and also have a kernel which has no continuous spectrum. For the multiparticle scattering problem, the Faddeev equations were generalized by many authors (Weinberg, 1964; Sugar & Blankenbecler, 1964; Rosenberg, 1965), who construct N-body integral equations with square integrable kernels. Their reformulated equations eliminate the δ-function singularities present in the kernel of the Lippmann–Schwinger equation (Lippman & Schwinger, 1950) due to disconnected processes. In that sense, the Fredholm theory is applicable. A significant advantage is introduced in Rosenberg's (1965) approach for the N-body problem since the potentials do not appear explicitly. Considering the Weinberg & Rosenberg generalizations, we introduced (Osman, 1970a) a cluster expansion in the light of the Faddeev equations.

In this paper, a cluster description of nuclei is developed in the framework of the Faddeev formalism, making use of previous considerations (Weinberg, 1964; Sugar & Blankenbecler, 1964; Rosenberg, 1965; Lippmann & Schwinger, 1950; Osman, 1970a; Lovelace, 1964a, 1964b). Composing the nuclei into smaller clusters, a set of generalized Faddeev equations are thus obtained to

apply to the many-cluster problem. Also, we considered the connectedness and the solvability of the obtained integral equations. The hard-cluster model (Wildermuth & Kanellopoulos, 1953, 1958/9; Phillips & Tombrello, 1960) and the cluster model inner parameters (Perlstein et al., 1960) approximations are discussed. Calculating the binding energies according to our model for the nuclei ^{6}Li, ^{8}Be, ^{12}C, ^{14}N, ^{16}O, and ^{20}Ne, using a different cluster decomposition for each, good agreement with the experimental measurements is obtained.

The model expansion with the generalized Faddeev equations for many-cluster problem are obtained in Sec. 2. The generalized Green's functions are defined in Sec. 3, while binding energy and generalized integral equations are introduced in Sec. 4. Results and calculations are introduced in Sec. 5. Section 6 is given over to discussion.

2. *The Model*

In a matrix form, Faddeev (1961a, 1961b, 1963) has represented the three-particle system successfully by his equations

$$\begin{pmatrix} T^{(1)}(Z) \\ T^{(2)}(Z) \\ T^{(3)}(Z) \end{pmatrix} = \begin{pmatrix} T_{23}(Z) \\ T_{31}(Z) \\ T_{12}(Z) \end{pmatrix} - \begin{pmatrix} 0 & T_{23}(Z) & T_{23}(Z) \\ T_{31}(Z) & 0 & T_{31}(Z) \\ T_{12}(Z) & T_{12}(Z) & 0 \end{pmatrix} G_0(Z) \begin{pmatrix} T^{(1)}(Z) \\ T^{(2)}(Z) \\ T^{(3)}(Z) \end{pmatrix} \quad (2.1)$$

where

$$T_{23}(Z) = V_{23} - V_{23} G_0(Z) T_{23}(Z) \quad (2.2)$$

$$G_0(Z) = (H_0 - Z)^{-1} \quad (2.3)$$

and the total t matrix $T(Z)$ is given by

$$T(Z) = T^{(1)}(Z) + T^{(2)} + T^{(3)}(Z) \quad (2.4)$$

with

$$T^{(1)}(Z) = V_{23} - V_{23} G_0(Z) T(Z) \cdots \quad (2.5)$$

and where V_{ij} is the two-body interaction between the particles i and j.

Let us suppose that the total system (nucleus) is to be divided into S subsystems. These subsystems are clusters denoted by S_n with $n = 1, 2, \ldots$. S refers to the cluster number. Thus, a ^{12}C nucleus can be divided by a method such as $^{12}\text{C} \rightarrow \alpha + \alpha + \alpha$, so that $S_1 = S_2 = S_3 = \alpha$ and the total system is divided into $S = 3$ subsystems.

If we start with the Faddeev equations (2.1), the N-body Lippmann–Schwinger equation for the S_n clusters can be written as

$$T(D(S)) = \sum_{i < D(S)} T(i; D(S)) \quad (2.6)$$

In Eq. (2.6) we have introduced the notation $D(S)$ to refer to the division process of the total system into S_n clusters with the corresponding number S classifying such a division, and with i being a pair of particles. In Eq. (6), for a division process $D(S)$, the amplitude is defined so that particles in a cluster S_n interact only with each other and not with particles in other clusters. This condition is clear and justified through the conditional notation $i < D(S)$.

Now, let the cluster S_n be subdivided by $D(C)$, so that no subsystem $D(C)$ is further divided by the division $D(S)$.

Thus in order that $D(C)$ cannot be divided further into subsystems, this condition is expressed by the notation $D(C) \leqslant D(S)$. To see this consider the above example of the ^{12}C nucleus, so that if $D(S)$ stands for $^{12}\mathrm{C} \to \alpha + \alpha + \alpha$, then more possible subdivision processes $D(C)$ are

$$^{12}\mathrm{C} \to \alpha + (d + d) + (d + d)$$

Therefore, we obtain the satisfied integral equation

$$T(i; D(S)) = T(i; D(C)) + T(i; D(C))G_0 \sum_{D(C) < j < D(S)} T(j; D(S)) \qquad (2.7)$$

where G_0 is the Green's function in free space and where we can find the pair j only in any subsystem of $D(S)$, and the notation $D(C) < j$ means that we cannot find the pair j in any subsystem of $D(C)$.

Let us introduce the vector $\mathbf{C}(S_n)$ with $\mathbf{S}$ components to classify the division $D(C)$ which follows $D(S)$. In this notation the kth component of $\mathbf{C}(S_n)$ refers to the number of clusters into which the kth subsystem of $D(S)$ is divided by $D(C)$. Thus taking the example of the ^{12}C nucleus, so that if $D(S)$ stands for $^{12}\mathrm{C} \to \alpha + \alpha + \alpha$, with a possible subdivision

$$^{12}\mathrm{C} \to \alpha + (d + d) + (n + n + p + p)$$

this subdivision is expressed as $\mathbf{C}(S_n) = (1, 2, 4)$.

It can be proved by mathematical induction that in a general form we can get

$$T(i; D(C)) = \sum_{\Upsilon(C) \geqslant \Upsilon_0(C)} M(i; D(\gamma)) + \sum_{\Upsilon(C) = \Upsilon_0(C)} M(i; D(\gamma))G_0 \sum_{D(\gamma) < j < D(C)} T(j; D(C)) \qquad (2.8)$$

where the vector $\mathbf{\Upsilon}_0(C)$ is a fixed $C(S_n)$-component vector, and the vector $\mathbf{C}(S_n)$ is introduced to classify the division $D(C)$ which follows $D(S)$. The notation $\mathbf{\Upsilon}(C) \geqslant \mathbf{\Upsilon}_0(C)$ means that each component of $\mathbf{\Upsilon}(C)$ is not smaller than the corresponding component of $\mathbf{\Upsilon}_0(C)$. By the equality $\mathbf{\Upsilon}(C) = \mathbf{\Upsilon}_0(C)$ we mean that every component of one is equal to that of the other. The quantity $M(i; D(\gamma))$ which appears in Eq. (2.8) is defined by Eq. (2.8) itself, since only when Eq. (2.8) holds are the $M(i; D(\gamma))$'s meaningful.

To prove Eq. (2.8), by mathematical induction, let us introduce Eq. (2.8) into Eq. (2.7) and sum over all $D(C) < D(S)$ for fixed $\mathbf{C}(S)$ and $\mathbf{\Upsilon}_0(S)$. The second condition gives some freedom for $\mathbf{\Upsilon}_0(S)$, since, for example, if we take

$\mathbf{C}(S) = (1, 2, 2)$ and $\boldsymbol{\Upsilon}_0(S) = (1, 3, 2)$, then two cases are possible, namely, $(1, 1, 2, 1, 1)$ and $(1, 2, 1, 1, 1)$. Then, we get with constant times

$$A(\mathbf{C}(S);\mathbf{i}(S))\,T(i;D(S)) = \sum_{\boldsymbol{\Upsilon}(S)\geqslant\boldsymbol{\Upsilon}_0(S)} [A(\mathbf{C}(S);\boldsymbol{\Upsilon}(S))\,M(i;D(\gamma)) + B(\mathbf{C}(S);\boldsymbol{\Upsilon}(S))M(i;D(\gamma))\,G_0 \times \sum_{D(\gamma)<j<D(S)} T(j;D(S))] \tag{2.9}$$

where

$$A(\mathbf{C}(S);\boldsymbol{\Upsilon}(S)) = \prod_{k=1}^{S_n} \frac{a(C_k(S);\gamma_k(S))}{C_k(S)!} \tag{2.10}$$

which denotes the number of possible $D(\gamma)$'s for a fixed $\mathbf{C}(S)$ and $\boldsymbol{\Upsilon}(S)$. Also, the recurrence relation defining $a(x;y)$ which appeared in eq. (2.10) is

$$a(1;y) = 1$$

$$a(x;y) = x^y - \binom{x}{x-1}\,a(x-1;y) - \binom{x}{x-2}a(x-2;y) - \cdots - \binom{x}{1}a(1;y) \tag{2.11}$$

In Eq. (9), we have also

$$B(\mathbf{C}(S);\boldsymbol{\Upsilon}(S)) = A(\mathbf{C}(S);\boldsymbol{\Upsilon}(S)) - \sum_{k=1}^{S_n} A(\mathbf{C}(S);\boldsymbol{\Upsilon}(S)-\mathbf{k}) \tag{2.12}$$

where $\mathbf{k}$ is the unit vector on the kth axis.

Thus, since we get Eq. (2.12), we see that Eq. (2.9) reduces to Eq. (2.8) with $D(C)$ replaced by $D(S)$. Now, since Eq. (2.8) exists, so the quantity $M(i, D(\gamma))$ is defined by Eq. (2.8) itself. For the connectedness of the model, Eq. (2.8) will be a fully connected equation only when

$$\gamma(S_n) = C(S_n) + 1$$

Also, approximating $M(D(S))$ by its poles (Osman 1970b) presents a solvable model for interacting several-particle systems by cluster deformation. Also Eq. (2.8) gives the possibility of decomposing clusters into smaller clusters, and so on. They are approximated by their lowest poles, which originate from the most highly connectected part of $T(D(S))$ and $M(D(S))$

3. *Generalized Green's Functions*

The T operator defined in Sect. 2 gives all the different physically relevant information wanted. As is shown from Eq. (2.8), the unperturbed Green's

function G_0 is given by Eq. (2.3), where H_0 is the sum of kinetic energy operators, so that if we suppose that the total system is composed of N particles, then

$$H_0 = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} \tag{3.1}$$

Then we can construct the full Green's function as

$$G(Z) = G_0(Z) + G_0(Z)T(Z)G_0(Z) \tag{3.2}$$

where

$$G(Z) = (H - Z)^{-1} \tag{3.3}$$

and where H is the Hamiltonian of the system.

From equations (2.3) and (3.3) we can expand

$$G(Z) = [1 + K(Z) + K^2(Z) + \cdots]G_0(Z) \tag{3.4}$$

where

$$K(Z) = G_0(Z)V \tag{3.5}$$

Any particle with binding energy ϵ shows up in $G(Z)$ as a pole at $Z = -\epsilon$ in the center-of-mass system and also generates cuts in Z with branch points at the thresholds for states containing the particle.

All the elements $T_{\alpha\beta}$ of Eq. (3.2) of the center-of-mass T matrix may be obtained using

$$T_{\alpha\beta}(\mathbf{k}_\alpha, \mathbf{k}_\beta; E) = \lim_{Z \to E + i\epsilon} [\langle \Phi_\alpha, (H - E)\Phi_\beta \rangle + \langle (H - E)\Phi_\alpha, G(Z)(H - E)\Phi_\beta \rangle] \tag{3.6}$$

where Φ_α is the plane wave describing the free relative motion of the bound subsystems in the channel α:

$$\Phi_\alpha(\boldsymbol{\rho}_\alpha, \mathbf{r}_\alpha) = X_\alpha(\boldsymbol{\rho}_\alpha)e^{i\mathbf{k}_\alpha \cdot \mathbf{r}_\alpha} \tag{3.7}$$

and $X_\alpha(\boldsymbol{\rho}_\alpha)$ is the bound-state wave functions.

Thus with the expression (3.5) for the scattering kernel, we can construct the full Green's function as

$$G(Z) = G_0(Z) + K(Z)G(Z) \tag{3.8}$$

The linear integral equation (3.8) is straightforward if the kernel $K(Z)$ is completely continuous. The kernel $K(Z)$ will be completely continuous if it is on the Hilbert-Schmidt kernel, that is if

$$\tau(Z) = \mathrm{Tr}[K(Z)K^\dagger(Z)] = \mathrm{Tr}[(1/|Z - H_0|^2)V^2] < \infty \tag{3.9}$$

One way to solve Eq. (3.8) is the modified Fredholm method. This method is based on the fact that if the kernel $K(Z)$ is completely continuous, then $G(Z)$ is a meromorphic function of the coupling constant, so that we can write

$$G(Z) = D^{-1}(Z)N(Z) \tag{3.10}$$

where the operators N and D are entire functions of the coupling constant. It is shown that N and D with $\mathrm{Tr}K^2$, $\mathrm{Tr}K^3$, . . ., etc. but not $\mathrm{Tr}K$ converge if $\tau(Z)$ is finite. This method had been given before in detail (Smithies, 1958). Then the binding energy can be computed by searching for the zeros of the Fredholm determinants.

Let us define the Green's function $G_S(Z)$ for any subsystem as

$$G_S(Z) = (H_S - Z)^{-1} \tag{3.11}$$

Then we can introduce the connected part of the Green's function as

$$C(Z) = I(Z)G(Z) \tag{3.12}$$

where $I(Z)$ is the sum of all irreducible ones.

This can be shown for the three-particle system if we define the partially connected Green's functions $P_{ij}(Z)$ as

$$G_{ij}(Z) = P_{ij}(Z) + G_0(Z) \tag{3.13}$$

with (for ij = 12, 13, 23)

$$G_{ij} = (H_0 + V_{ij} - Z)^{-1} \tag{3.14}$$

Thus the completely connected Green's function is

$$G(Z) = C(Z) + P_{12}(Z) + P_{13}(Z) + P_{23}(Z) + G_0(Z) \tag{3.15}$$

with

$$P_{ij}(Z) = K_{ij}(Z)G_{ij}(Z)$$

and then the irreducible kernel $I(Z)$ is defined by

$$I(Z) = P_{12}(Z)\{V_{23} + V_{13}\} + P_{23}(Z)\{V_{12} + V_{13}\} + P_{13}(Z)\{V_{12} + V_{23}\} \tag{3.16}$$

So, that we can obtain

$$C(Z) = B(Z) + I(Z)C(Z) \tag{3.17}$$

The expression (3.17) can be generalized for any subsystem S in an N-particle system as

$$C_S(Z) = B_S(Z) + I_S(Z)C_S(Z) \tag{3.18}$$

If the number of particles in S is 4, for example, then for the connected part we have

$$\begin{aligned} C_{ijkl} = {} & (H_{ijkl} - Z)^{-1} - (H_{ijk} + H_l - Z)^{-1} - (H_{ijl} + H_k - Z)^{-1} \\ & - (H_{ikl} + H_j - Z)^{-1} - (H_{jkl} + H_i - Z)^{-1} - (H_{ij} + H_{kl} - Z)^{-1} \\ & - (H_{ik} + H_{jl} - Z)^{-1} - (H_{il} + H_{jk} - Z)^{-1} \\ & + 2(H_{ij} + H_k + H_l - Z)^{-1} + 2(H_{ik} + H_j + H_l - Z)^{-1} \\ & + 2(H_{il} + H_j + H_k - Z)^{-1} + 2(H_{jk} + H_i + H_l - Z)^{-1} \\ & + 2(H_{jl} + H_i + H_k - Z)^{-1} + 2(H_{kl} + H_i + H_j - Z)^{-1} \\ & - 6(H_i + H_j + H_k + H_l - Z)^{-1} \end{aligned} \tag{3.19}$$

For the irreducible kernel we have

$$\begin{aligned} I_{ijkl} = {} & [C_{ijk} * C_l](V_{il} + V_{jl} + V_{kl}) + [C_{ijl} * C_k](V_{ik} + V_{jk} + V_{lk}) \\ & + [C_{ikl} * C_j](V_{ij} + V_{kj} + V_{lj}) + [C_{jkl} * C_i](V_{ji} + V_{ki} + V_{li}) \\ & + [C_{ij} * C_{kl}](V_{ik} + V_{il} + V_{jk} + V_{jl}) + [C_{ik} * C_{jl}](V_{ij} + V_{il} + V_{kj} + V_{kl}) \\ & + [C_{il} * C_{jk}](V_{ij} + V_{ik} + V_{lj} + V_{lk}) \end{aligned} \tag{3.20}$$

with expressions for C_{ijk}, I_{ijk} and C_{ij}, I_{ij} more simple than expressions (3.19) and (3.20) describing the three- and two-particle systems.

Thus the bound state of the subsystem S occurs when the connected Green's function for that S has a pole. Such poles arise when one of the eigenvalues of the corresponding irreducible kernel passes unity.

4. *Binding Energy and Generalized Integral Equations*

To write now the explicit forms of the generalized integral equations defining the quantities which appeared in Sect. 3, we start by the two-body problem.

For nonrelativistic two-particle scattering, the kernel is

$$\langle \mathbf{p}_1 \mathbf{p}_2 | K(Z) | \mathbf{p}_1' \mathbf{p}_2' \rangle = \frac{\delta(\mathbf{P} - \mathbf{P}')}{Z - \mathbf{q}^2/2\mu - \mathbf{P}^2/2M} \langle \mathbf{q} | V_{12} | \mathbf{q}' \rangle \tag{4.1}$$

where

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

$$\mathbf{q} = (m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2)/(m_1 + m_2)$$

$$M = m_1 + m_2$$

and

$$\mu = m_1 m_2/(m_1 + m_2)$$

To avoid the δ function that appeared in Eq. (4.1), preventing it from being an L^2 kernel, i.e., $\tau(Z) < \infty$, we have to define

$$\langle \mathbf{p}_1\mathbf{p}_2|G(Z)|\mathbf{p}_1'\mathbf{p}_2'\rangle \equiv \delta(\mathbf{P}-\mathbf{P}')\,\langle \mathbf{q}|G(Z)-\mathbf{P}^2/2M|\mathbf{q}'\rangle \tag{4.2}$$

Introducing Eq. (4.2) into Eq. (3.8), we get

$$\langle \mathbf{q}|G(Z)|\mathbf{q}'\rangle = \delta(\mathbf{q}-\mathbf{q}')/(Z-\mathbf{q}^2/2\mu) + \int d^3q''\,\frac{\langle \mathbf{q}|V_{12}|\mathbf{q}''\rangle\langle \mathbf{q}''|G(Z)|\mathbf{q}'\rangle}{Z-\mathbf{q}^2/2\mu} \tag{4.3}$$

The integral equation (4.3) is an L^2 kernel only if we have

$$\int d^3q\,d^3q'\,\frac{|\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle|^2}{|Z-\mathbf{q}^2/2\mu|^2} < \infty \tag{4.4}$$

For local potentials $V_{12}(\mathbf{r})$, the condition (4.4) is

$$\int d^3r V_{12}^2(\mathbf{r}) < \infty \tag{4.5}$$

which is true for short-range potentials. Then if $V_{12}(\mathbf{r})$ is a local potential, for every partial wave the kernel will be an L^2 kernel if

$$\int_0 r^2 V_{12}^2(r)dr < \infty \quad \text{and} \quad \int^{\infty} V_{12}^2(r)dr < \infty \tag{4.6}$$

Now, let us turn to the more complicated problems of an N-particle system, considering firstly the easiest one of the three distinguishable particles with two-body interactions V_{ij}.

In that case the matrix elements of the kernel will be written as

$$\begin{aligned}\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|K(Z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \frac{\delta(\Sigma_i\mathbf{p}_i - \Sigma_i\mathbf{p}_i')}{Z-\Sigma_i\mathbf{p}_i^2/2m_i}\,[&\delta(\mathbf{p}_3-\mathbf{p}_3')\,\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle \\ &+ \delta(\mathbf{p}_1-\mathbf{p}_1')\langle \mathbf{q}_{23}|V_{23}|\mathbf{q}_{23}'\rangle \\ &+ \delta(\mathbf{p}_2-\mathbf{p}_2')\langle \mathbf{q}_{13}|V_{13}|\mathbf{q}_{13}'\rangle]\end{aligned} \tag{4.7}$$

Then from Eqs. (3.13–3.16), we have the integral equations.

$$P_{ij}(Z) = K_{ij}(Z)G_0(Z) + K_{ij}(Z)P_{ij}(Z) \tag{4.8}$$

and for the inhomogeneous term $B(Z)$ that appeared in Eq. (3.17)

$$B(Z) = I(Z)[P_{12}(Z) + P_{13}(Z) + P_{23}(Z) + G_0(Z)] \tag{4.9}$$

Now for P_{12} in Eq. (4.8), the kernel of the integral equation is

$$\begin{aligned}\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|K_{12}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{p}_3-\mathbf{p}_3') \\ \times \frac{\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle}{[Z-\mathbf{p}_3^2/2\mu_3 - \mathbf{q}_{12}^2/2\mu_{12}]}\end{aligned} \tag{4.10}$$

where we have used

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$$
$$\mathbf{q}_{12} \equiv (m_2\mathbf{q}_1 - m_1\mathbf{q}_2)/(m_1 + m_2)$$
$$\mu_{12} \equiv m_1 m_2/(m_1 + m_2)$$

and

$$\mu_3 = m_3(m_1 + m_2)/(m_1 + m_2 + m_3)$$

Then Eq. (4.8) has the solution

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|P_{12}(Z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3')\delta(\mathbf{p}_3 - \mathbf{p}_3') \times \langle \mathbf{q}_{12}|P_{12}(Z\mathbf{p}_3^2/2\mu_3)|\mathbf{q}_{12}'\rangle \tag{4.11}$$

with an integral equation for the reduced matrix element given as

$$\langle \mathbf{q}|P_{12}(Z)|\mathbf{q}'\rangle = \frac{\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle}{(Z - \mathbf{q}^2/2\mu_{12})(Z - \mathbf{q}'^2/2\mu_{12})} + \int d^3q'' \frac{\langle \mathbf{q}|V_{12}|\mathbf{q}''\rangle\langle \mathbf{q}''|P_{12}(Z)|\mathbf{q}'\rangle}{Z - \mathbf{q}^2/2\mu_{12}} \tag{4.12}$$

From Eq. (4.12) it is clear that $\langle \mathbf{q}|P_{12}(Z)|\mathbf{q}'\rangle$ is the two-body T matrix which can be solved by the Fredholm method if the potential V_{12} is chosen to satisfy the condition

$$\int \frac{|\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle|^2}{|Z - \mathbf{q}^2/2\mu_{12}|^2} d^3q d^3q' < \infty \tag{4.13}$$

Under the condition (4.13), $P_{12}(Z)$ as given by the integral equation (4.12) can be solved, and similarly for the quantities $P_{23}(Z)$ and $P_{13}(Z)$.

Let us now consider the integral for $C(Z)$ expressed by Eq. (3.17), where the first term is given by

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|P_{12}(Z)V_{23}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3')\langle \mathbf{q}_{12}|P_{12}(Z - \mathbf{p}_3^2/2\mu_3)|\mathbf{q}_{12}'\rangle \times \langle \mathbf{q}_{23}''|V_{23}|\mathbf{q}_{23}'\rangle \tag{4.14}$$

with

$$\mathbf{q}_{12}' = \frac{m_2\mathbf{p}_1' - m_1\mathbf{p}_2'}{m_1 + m_2} = \mathbf{p}_1' + \frac{m_1}{m_1 + m_2}\mathbf{p}_3$$

$$\mathbf{q}_{23}'' = \frac{m_3\mathbf{p}_2' - m_2\mathbf{p}_3}{m_2 + m_3} = -\mathbf{p}_3 - \frac{m_3}{m_2 + m_3}\mathbf{p}_1'$$

and

$$\mathbf{p}_2' = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_1' = -\mathbf{p}_3 - \mathbf{p}_1'$$

Using Eq. (4.14), all the terms of Eq. (3.17) can be obtained by simple permutations, and also the inhomogeneous term $B(Z)$ can be obtained from Eq. (4.9). Finally, since $I(Z)$ is an L^2 kernel, we can solve equation (3.17) for $C(Z)$. Then $I(Z)$ can be written as

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I(Z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3') \times \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I_{\text{c.m.}}(Z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle \tag{4.15}$$

where $I_{\text{c.m.}}$ does not contain any δ function. Then $I(Z)$ will be an L^2 kernel if the center-of-mass Hilbert–Schmidt norm $\|I(Z)\|_2 < \infty$, expressed as

$$\begin{aligned}\|I(Z)\|_2 \equiv \Big[& \int d^3p_1 d^3p_2 d^3p_3 \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \\ & \times \int d^3p_1' d^3p_2' d^3p_3' \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3') \\ & \times |\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I_{\text{c.m.}}(z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle|^2 \Big]^{1/2}\end{aligned} \tag{4.16}$$

Then according to the condition (4.13) $\|I(Z)\|_2$ is finite subject to this condition (4.13) for all V_{ij}.

In Eq. (4.16) we have

$$\|K_{ij}(Z)K_{jk}(Z)\|_2 < \infty \quad (ijk = 123, 231, 312) \tag{4.17}$$

Any bound state with energy $-\epsilon < 0$ must correspond to a pole of $T(Z)$ at $Z = -\epsilon$. In case the V are weak, the integral equation for $T(Z)$ could be solved by perturbation theory to give the Born series, in which no term has any poles. Thus the bound state only exists if V is strong enough so that the series converge at least for Z in the neighborhood of $-\epsilon$.

For local potentials V_{ij}, (4.16) and (4.17) will be satisfied if

$$\int d^3r |V_{ij}(\mathbf{r})|^2 < \infty \quad (ij = 12, 23, 13) \tag{4.18}$$

Considering that particles 12 and 23 form bound states with binding energies ϵ_{12} and ϵ_{23}, the pole approximation is suggested by Faddeev (1961a, 1961b, 1963) and Lovelace (1964a, 1964b) and has been applied by Rosenberg (1965) introduced as

$$\begin{aligned}\langle \mathbf{q}_{12}|P_{12}(Z)|\mathbf{q}_{12}'\rangle &\simeq \frac{\psi_{12}(\mathbf{q}_{12})\overset{*}{\psi}_{12}(\mathbf{q}_{12}')}{Z + \epsilon_{12}} \\ \langle \mathbf{q}_{23}|P_{23}(Z)|\mathbf{q}_{23}'\rangle &\simeq \frac{\psi_{23}(\mathbf{q}_{23})\overset{*}{\psi}_{23}(\mathbf{q}_{23}')}{Z + \epsilon_{23}} \\ \langle \mathbf{q}_{13}|P_{13}(Z)|\mathbf{q}_{13}'\rangle &\simeq 0\end{aligned} \tag{4.19}$$

Then the irreducible center-of-mass kernel for three-particle systems according to this approximation, making use of Eqs. (3.16), (4.11), and (4.14), can be

Figure 1. The sum $C(Z)$ of all connected graphs.

written as

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 | I_{\text{c.m.}}(Z) | \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3' \rangle = \frac{\psi_{12}(q_{12})\overset{*}{\psi}_{12}(q_{12}')[V_{31}(\mathbf{p}_3 - \mathbf{p}_3') + V_{32}(\mathbf{p}_3 - \mathbf{p}_3')]}{Z + \epsilon_{12} - \mathbf{p}_3^2/2\mu_3}$$
$$+ \frac{\psi_{23}(q_{23})\overset{*}{\psi}_{23}(q_{23}')[V_{12}(\mathbf{p}_1 - \mathbf{p}_1') + V_{13}(\mathbf{p}_1 - \mathbf{p}_1')]}{Z + \epsilon_{23} - \mathbf{p}_1^2/2\mu_1} \tag{4.20}$$

Thus, solving the kernel (4.20), which is an L^2 kernel, by the Fredholm method we can calculate the bound-state binding of the three-body problems by solving the eigenvalue equation

$$I(-\epsilon_{123})\Psi = \Psi \tag{4.21}$$

Introducing Eq. (4.20) into the integral equation (3.17), and assuming that the overlap integrals for ψ_{12} and ψ_{23} can be treated in first order, the well-known distorted wave approximation is obtained.

Now, for more complicated N-particle systems, as for example $N = 4$, the integral equations for the four-body systems are expressed in Figs. 1, 2, and 3, corresponding to Eqs. (3.13) and (3.15), Eqs. (3.17) and (3.18), and Eqs. (3.17) and (3.18), respectively.

Solving these equations successively, the explicit expressions for C_S and I_S are given by Eqs. (3.19) and (3.20) for $N = 4$.

Thus it is possible to solve the N-cluster problem in a lengthy but straightforward way whatever the number of cluster decompositions N. First, we solve the connected Green's function using the ordinary Lippmann-Schwinger integral equation (Figs. 2 and 3, the first parts). Then we solve the irreducible

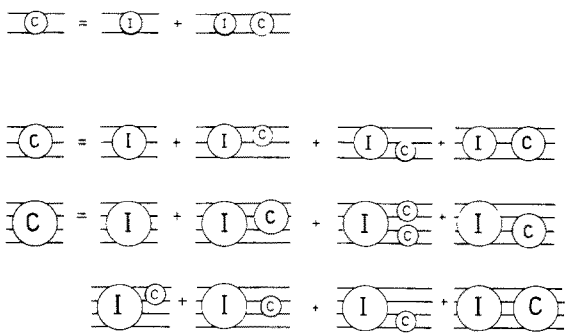

Figure 2. The linear integral equations for the connected Green's functions for $N = 2$, 3, and 4 clusters.

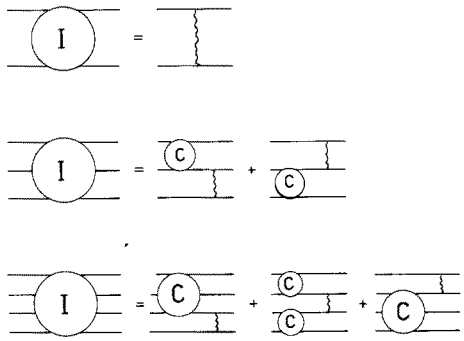

Figure 3. The irreducible kernels $I(Z)$ appearing in the integral equations of $C(Z)$ for 2, 3, and 4 clusters.

kernel for the three-cluster problem (Fig. 3, the second part) and the inhomogeneous term (Fig. 2, the second part) using the Fredholm method

In the cluster equation, the partially connected equation is in general not solvable. With the approximations introduced and continuing the connecting process, the fully connected equations are obtained. In principle, one can construct more highly connected kernels by solving the fully connected equations step by step.

For any system, the bound state occurs when the connected Green's function for that system has a pole. These poles arise when one of the eigenvalues of the corresponding irreducible kernel passes unity [Fig. 2 and Eqs. (3.17) and (3.18)].

If we are interested in the channel α, for the case of subsystems $S_1, S_2, \ldots, S_m$ bound together with total eigenvalue $-\epsilon_\alpha$, then the total energy is $E = S_\alpha + (\hbar k_\alpha^f)^2/2m_\alpha$. This decomposition process may continue till a class of terms contain as their left-hand factor the operator $M(S_1; S_2; \ldots; S_m)$ which leads to the channel of interest. This class of terms contributes to the residue of a pole at $-\epsilon_\alpha$ of $M(D(S))$ with energy $E' = -\epsilon_\alpha + (\hbar k'_\alpha)^2/2m_\alpha$. Therefore the existence of the pole is deduced by comparing E and E', where from the momentum conservation the matrix element of $M(D(S))$ vanishes unless $\mathbf{k}'_\alpha = \mathbf{k}^f_\alpha$. Thus the residue is taken at the pole at $S_\alpha = -\epsilon_\alpha$ in the momentum-space matrix element of $M(D(S))$. Usually, the residue at a bound-state pole factorizes. In the neighborhood of a bound state, the residue of the off-shell two-body amplitude factorizes in the initial and final moments. The resulting functions of the momenta are the form factors. These form factors, in the bound-state case, are simply related to the bound-state wave function. At these bound-state poles, the residues factorize in the off-shell momentum variables. In the case of a degenerate bound state, the residue will be a sum of separable terms.

It is worthwhile that we must mention here that our $M(D(S))$'s are identical to the $T^{(C)}$'s of the Rosenberg equation.

5. *Results and Calculations*

Our model developed in Sec. 2 is applied for practical numerical calculations. The binding energies for the nuclei ^{6}Li, ^{8}Be, ^{12}C, ^{14}N, ^{16}O and ^{20}Ne are calculated. Different cluster deformation for these nuclei is suggested. Thus the total t matrix of a nucleus is expressed as an expansion of sum of t matrices according to the cluster model required for any nucleus. Once the total t matrix is well defined, the binding energy can be extracted as a method presented for the three-nucleon system (Osman, 1970b), and more complicated systems (Osman, 1970c, 1971), and is generalized for the present model. Equation (2.8) gives the possibility of decomposing clusters into smaller clusters, and so on. All of them are approximated by their lowest poles, which originate from the most highly connected part of $T(D(S))$ and $M(D(S))$.

Then, for numerical calculations of the Faddeev equations we have followed a method for direct solution of these equations. If we consider that L is the maximum orbital angular momentum in the two-body partial wave expansion, Ahmadzadeh and Tjon (1965) and Osborn and Noyes (1966) have shown that the Faddeev equations are reduced to a coupled set of $3(L + 1)$ integral equations in two continuous variables for the $J = 0$ problem. The number of these integral equations is reduced to $(L + 1)$ in the case of three identical particles. Osborn (1967) has introduced the approximation of only considering the $L = 0$ part of the potential, so that the three-body problem is described by one integral equation in two variables. For separable potentials, the integral equation may be reduced to an integral equation in one variable, which by conformal mapping and Gaussian quadrature is converted into a sum. For nonseparable potentials, special quadrature rules by which a small number of mesh points give accurate results have been developed. In our present work the integrals are solved using the Kopal (1955) method. These integrals are replaced by 20-45 point mesh. The number of mesh points considered is taken according to every cluster decomposition in every case. In the integration, the kernel is approximated by a finite $N \times N$ matrix by choosing finite mesh sizes. The number of steps N in the integration is taken to be 45. Increasing this number to $N = 75$ did not change the result.

Coulomb effects between the different clusters composing these nuclei have been included by a manner described in detail in Osman (1971). The results of the calculations of these nuclei are represented on Tables 1-6, respectively. For the purpose of comparison, the deviation of the present model calculations from the experimental measurements of the atomic masses (Mattauch et al., 1965; Endt & Van der Leun, 1967) is introduced, which as shown does not exceed about 2.767%. Also the deviation of the binding energy per nucleon between calculations of our present model and experimental binding energy per nucleon is found that it does not exceed about 0.11045 MeV.

These obtained generalized Faddeev equations in their limit case give the three-nucleon problem which is studied explicitly in Osman (1970c), using the separable expansion for the t matrix.

TABLE 1. The binding energy (B.E.) of the ^{6}Li nucleus in MeV

Model	B.E. due to our model	B.E. (Expt.)	Difference MeV	Deviation percent
$\alpha + d$	1.472	1.4723	0.0003	0.021
$\alpha + n + p$	3.691	3.697	0.006	0.162
$d + d + d$	25.419	25.319	0.100	0.394
$2d + n + p$	27.443	27.544	0.101	0.367
$d + 2n + 2p$	29.657	29.768	0.111	0.374
$3n + 3p$	32.012	31.993	0.019	0.059

TABLE 2. The binding energy (B.E.) of the ^{8}Be nucleus in MeV

Model	B.E. due to our model	B.E. (expt.)	Difference, MeV	Deviation, percent
$\alpha + \alpha$	0.126			
$\alpha + d + d$	23.398	23.752	0.354	1.502
$\alpha + d + n + p$	25.658	25.977	0.319	1.236
$\alpha + 2n + 2p$	28.324	28.201	0.123	0.435
$d + d + d + d$	47.333	47.599	0.266	0.560
$3d + n + p$	49.484	49.824	0.340	0.685
$d + d + 2n + 2p$	52.243	52.048	0.195	0.374
$d + 3n + 3p$	54.618	54.273	0.345	0.634
$4n + 4p$	56.221	56.498	0.277	0.492

TABLE 3. The binding energy (B.E.) of the ^{12}C nucleus in MeV

Model	B.E. due to our model	B.E. (expt.)	Difference, MeV	Deviation, percent
$\alpha + \alpha + \alpha$	7.475	7.274	0.201	2.726
$\alpha + \alpha + d + d$	30.957	31.121	0.164	0.528
$\alpha + 4d$	55.137	54.968	0.169	0.307
$2\alpha + d + n + p$	33.175	33.346	0.171	0.514
$\alpha + 3d + n + p$	56.993	57.193	0.200	0.350
$\alpha + \alpha + 2n + 2p$	35.489	35.570	0.081	0.228
$\alpha + 2d + 2n + 2p$	59.169	59.417	0.248	0.418
$\alpha + d + 3n + 3p$	61.856	61.642	0.214	0.347
$\alpha + 4n + 4p$	63.341	63.866	0.525	0.825
$6d$	78.418	78.815	0.397	0.505
$5d + n + p$	81.102	81.040	0.062	0.077
$4d + 2n + 2p$	83.014	83.285	0.271	0.326
$3d + 3n + 3p$	85.306	85.489	0.183	0.214
$2d + 4n + 4p$	87.482	87.714	0.232	0.265
$d + 5n + 5p$	89.888	89.938	0.050	0.056
$6n + 6p$	92.004	92.163	0.159	0.173

TABLE 4. The binding energy (B.E.) of the ^{14}N nucleus in MeV

Model	B.E. due to our model	B.E. (expt.)	Difference, MeV	Deviation, percent
$\alpha + \alpha + \alpha + d$	17.831	17.546	0.285	1.611
$\alpha + \alpha + \alpha + n + p$	19.463	19.771	0.308	1.570
$\alpha + \alpha + 3d$	41.193	41.394	0.201	0.487
$2\alpha + 2d + n + p$	43.405	43.618	0.213	0.490
$\alpha + 5d$	65.723	65.241	0.482	0.736
$\alpha + 4d + n + p$	67.852	67.465	0.387	0.572
$2\alpha + d + 2n + 2p$	45.231	45.843	0.612	1.344
$\alpha + 3d + 2n + 2p$	69.041	69.690	0.649	0.936
$\alpha + \alpha + 3n + 3p$	48.943	48.067	0.876	1.806
$\alpha + 2d + 3n + 3p$	72.237	71.914	0.323	0.448
$\alpha + d + 4n + 4p$	73.993	74.139	0.146	0.197
$\alpha + 5n + 5p$	76.853	76.363	0.490	0.640
$7d$	89.435	89.088	0.347	0.389
$6d + n + p$	91.068	91.312	0.244	0.268
$5d + 2n + 2p$	93.794	93.537	0.257	0.274
$4d + 3n + 3p$	95.274	95.761	0.487	0.510
$3d + 4n + 4p$	97.621	97.986	0.365	0.373
$2d + 5n + 5p$	99.897	100.210	0.313	0.313
$d + 6n + 6p$	102.548	102.435	0.113	0.110
$7n + 7p$	104.593	104.659	0.066	0.063

TABLE 5. The binding energy (B.E.) of the ^{16}O nucleus in MeV

Model	B.E. due to our model	B.E. (expt.)	Difference, MeV	Deviation, percent
$\alpha + \alpha + \alpha + \alpha$	14.042	14.436	0.394	2.767
$3\alpha + 2d$	37.823	38.283	0.540	1.419
$3\alpha + d + n + p$	40.801	40.507	0.294	0.723
$3\alpha + 2n + 2p$	42.321	42.732	0.411	0.967
$2\alpha + 4d$	61.786	62.130	0.344	0.555
$2\alpha + 3d + n + p$	64.575	64.354	0.221	0.343
$2\alpha + 2d + 2n + 2p$	66.954	66.579	0.375	0.562
$2\alpha + d + 3n + 3p$	68.173	68.803	0.630	0.920
$2\alpha + 4n + 4p$	70.732	71.028	0.296	0.418
$\alpha + 6d$	85.154	85.977	0.823	0.962
$\alpha + 5d + n + p$	88.063	88.201	0.138	0.157
$\alpha + 4d + 2n + 2p$	90.167	90.426	0.259	0.287
$\alpha + 3d + 3n + 3p$	92.889	92.650	0.239	0.258
$\alpha + 2d + 4n + 4p$	94.882	94.875	0.007	0.007
$\alpha + d + 5n + 5p$	96.860	97.093	0.233	0.240
$\alpha + 6n + 6p$	99.142	99.324	0.182	0.183
$8d$	108.128	109.824	0.696	0.639
$7d + n + p$	111.243	112.048	0.805	0.721
$6d + 2n + 2p$	113.899	114.273	0.374	0.328
$5d + 3n + 3p$	116.217	116.497	0.280	0.241
$4d + 4n + 4p$	118.341	118.722	0.381	0.321
$3d + 5n + 5p$	120.567	120.946	0.379	0.314
$2d + 6n + 6p$	122.894	123.171	0.277	0.225
$d + 7n + 7p$	125.163	125.395	0.232	0.185
$8n + 8p$	127.352	127.620	0.268	0.210

TABLE 6. The binding energy (B.E.) of the ^{20}Ne nucleus in MeV

Model	B.E. due to our model	B.E. (expt.)	Difference, MeV	Deviation, percent
5α	18.995	19.166	0.171	0.897
$4\alpha + 2d$	42.726	43.012	0.286	0.667
$4\alpha + d + n + p$	44.884	45.237	0.353	0.783
$4\alpha + 2n + 2p$	47.769	47.461	0.308	0.647
$3\alpha + 4d$	66.126	66.859	0.733	1.102
$3\alpha + 3d + n + p$	68.279	69.084	0.805	1.172
$3\alpha + 2d + 2n + 2p$	71.487	71.278	0.209	0.293
$3\alpha + d + 3n + 3p$	73.146	73.533	0.387	0.528
$3\alpha + 4n + 4p$	75.254	75.757	0.503	0.666
$2\alpha + 6d$	89.876	90.706	0.830	0.919
$2\alpha + 5d + n + p$	92.135	92.931	0.796	0.860
$2\alpha + 4d + 2n + 2p$	94.489	95.156	0.667	0.703
$2\alpha + 3d + 3n + 3p$	96.852	97.380	0.528	0.544
$2\alpha + 2d + 4n + 4p$	98.791	99.605	0.814	0.821
$2\alpha + d + 5n + 5p$	101.031	101.829	0.798	0.787
$2\alpha + 6n + 6p$	103.759	104.054	0.295	0.284
$\alpha + 8d$	113.138	114.554	1.416	1.244
$\alpha + 7d + n + p$	115.974	116.778	0.804	0.691
$\alpha + 6d + 2n + 2p$	117.907	119.003	1.096	0.966
$\alpha + 5d + 3n + 3p$	120.286	121.227	0.941	0.779
$\alpha + 4d + 4n + 4p$	122.701	123.452	0.751	0.610
$\alpha + 3d + 5n + 5p$	124.978	125.676	0.698	0.557
$\alpha + 2d + 6n + 6p$	127.058	127.901	0.843	0.661
$\alpha + d + 7n + 7p$	129.249	130.125	0.876	0.676
$\alpha + 8n + 8p$	132.405	132.350	0.055	0.042
$10d$	136.192	138.401	2.209	1.609
$9d + n + p$	139.439	140.625	1.186	0.847
$8d + 2n + 2p$	141.295	142.850	1.555	1.095
$7d + 3n + 3p$	143.761	145.074	1.313	0.909
$6d + 4n + 4p$	146.082	147.299	1.217	0.837
$5d + 5n + 5p$	148.345	149.523	1.178	0.791
$4d + 6n + 6p$	150.618	151.748	1.130	0.747
$3d + 7n + 7p$	152.903	153.972	1.069	0.697
$2d + 8n + 8p$	155.432	156.197	0.765	0.491
$d + 9n + 9p$	157.721	158.421	0.700	0.443
$10n + 10p$	159.875	160.646	0.771	0.481

6. *Discussion*

From Sec. 2, it is clear that according to our present developed model, the hard-cluster model (Wildermuth & Kanellopoulos, 1958/9; Phillips & Tombrella, 1960) and the cluster model inner parameters (Pearlstein et al., 1960) can be reconsidered using Eq. (2.8). For the hard-cluster model, the clusters are approximated by their poles originating from $T(D(S))$ and $M(D(S))$. Thus, approximating the $M(D(S))$'s by their poles together with renormalization is equivalent to projection upon the cluster states.

From Eq. (2.8) we have the possibility of decomposing clusters into smaller clusters. The Coulomb effects between the different clusters are found (Osman, 1971) to give accurate measured physical quantities in good agreement with the observed data.

From Tables 1–6, it is clear that the mode we have developed extracts well exactly the binding energy for nuclei which are not far from previous measured values. A glance at these tables shows that nucleons inside nuclei like to cluster in the form of α clusters.

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